

Markov Models: Examples

☉ Example 1: Matrix→Diagram, Equilibrium

Draw the system diagram for this matrix of transition probabilities:

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

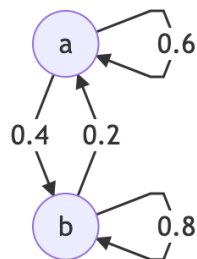
First let's add some labels to the matrix to remind ourselves what the entries mean:

$$\begin{array}{cc} & \begin{array}{c} \textit{To} \\ a \quad b \end{array} \\ \begin{array}{c} \textit{From} \\ a \\ b \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

So we can see we'll need two vertices,



And then we add the edges between them with the weights from the matrix:



Find the equilibrium state of the system¹:

The key here is that at equilibrium applying the transition matrix doesn't change the distribution of values so that the state before and after the transition is unchanged. In matrix multiplication terms we have:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

Carrying out the matrix multiplication gives us,

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \rightarrow \begin{cases} 0.6a + 0.2b = a \\ 0.4a + 0.8b = b \end{cases}$$

Two equations in two unknowns! Let's solve them:

$$\begin{aligned} 0.6a + 0.2b &= a &\rightarrow &-0.4a + 0.2b = 0 \\ 0.4a + 0.8b &= b &\rightarrow &0.4a - 0.2b = 0 \end{aligned}$$

Hmmm...If we add these we get $0=0$, which is true but not helpful, so these equations are colinear².

But it's not a dead end. Let's rearrange the first equation,

$$\begin{aligned} -0.4a + 0.2b &= 0 \\ 0.2b &= 0.4a \\ b &= \frac{0.4}{0.2}a \\ b &= 2a \end{aligned}$$

This tells us that at equilibrium the value of b will be twice the value of a . Well since b and a have to add to 100% we know that,

$$\begin{aligned} a + b &= 100\% \\ a + 2a &= 100\% \\ 3a &= 100\% \\ a &\approx 33\% \text{ and } b = 2a \approx 66\% \end{aligned}$$

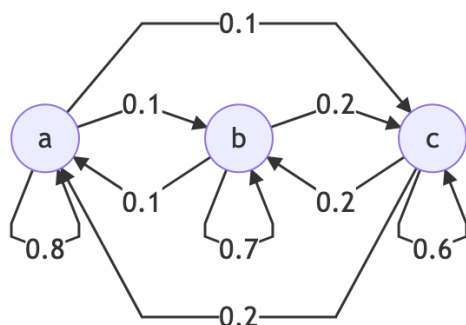
In percentage terms this means that b will be 66% and a will be 33% of whatever our total amount in the system is.

¹ The equilibrium state is sometimes called the *stationary distribution* because like a stationary point on a function, when the system achieves the stationary distribution of values, it does not move from it.

² Why are they colinear? Remember that the probabilities are not independent. If we know all but one probability leaving the vertex we can find the missing one, so there is some redundancy in our system. That shows up here.

⊙ Example 2: Diagram→Matrix, Equilibrium

Find the transition matrix and equilibrium state for this system.



We see the diagram has three vertices so we'll need a 3x3 transition matrix,

$$\begin{matrix} & a & b & c \\ \begin{matrix} a \\ b \\ c \end{matrix} & \left[\begin{array}{ccc} & & \end{array} \right] \end{matrix}$$

Then we enter the edge values into the matrix remembering that edges are entered in the row they are coming *from* and the column they are going *to*.

$$\begin{matrix} & a & b & c \\ \begin{matrix} a \\ b \\ c \end{matrix} & \left[\begin{array}{ccc} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{array} \right] \end{matrix}$$

What about this system's equilibrium value? As before we know that the before transition and after transition state vectors will be the same,

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$$

Which gives us the equations,

$$\begin{aligned} 0.8a + 0.1b + 0.2c &= a \\ 0.1a + 0.7b + 0.2c &= b \\ 0.1a + 0.2b + 0.6c &= c \end{aligned}$$

Gathering the variables on the LHS,

$$\begin{aligned} -0.2a + 0.1b + 0.2c &= 0 \\ 0.1a - 0.3b + 0.2c &= 0 \\ 0.1a + 0.2b - 0.4c &= 0 \end{aligned}$$

Solving this gives³,

$$\begin{aligned}a - 1.6c &= 0 \\b - 1.2c &= 0 \\0 &= 0\end{aligned}$$

What does this mean? We aren't getting unique values for each of a , b and c . True, but we are getting a unique distribution. These equations tell us that at equilibrium the value of b will be 1.2 times as large as c , and that the value of a will be 1.6 times as large as c . So,

If we want to express the state in %s across a , b and c this means that,

$$\begin{aligned}1.6c + 1.2c + c &= 100 \\3.8c &= 100 \\c &\approx 26.32\end{aligned}$$

And,

$$\begin{aligned}b &= 1.2c \approx 31.58 \\a &= 1.6c \approx 42.11\end{aligned}$$

What does Excel say happens? Starting from an initial even distribution across a , b and c we get,

			a	0.8	0.1	0.1	0.8	0.1	0.1	0.8	0.1	0.1	0.8	0.1	0.1	0.8	0.1	0.1	0.8	0.1	0.1		
			b	0.1	0.7	0.2	0.1	0.7	0.2	0.1	0.7	0.2	0.1	0.7	0.2	0.1	0.7	0.2	0.1	0.7	0.2		
			c	0.2	0.2	0.6	0.2	0.2	0.6	0.2	0.2	0.6	0.2	0.2	0.6	0.2	0.2	0.6	0.2	0.2	0.6		
a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
33	33	33	36	33	30	38	33	28	40	32	27	40	32	27	41	32	26	41	32	26	41	32	26

³ Results might appear to vary depending on the order in which you solve the equations, but will be equivalent to this. If you make substitutions in a different order you might get the equations

A more direct way to get the percentages

We can get the percentage distribution as part of solving our system of equations if we add a fourth equation to the mix,

$$a + b + c = 100$$

This just adds the requirement that a , b and c have to add to 100%.

Solving the resulting system of 4 equations in 3 unknowns using a matrix gives us

$$\begin{bmatrix} -0.2 & 0.1 & 0.2 & 0 \\ 0.1 & -0.3 & 0.2 & 0 \\ 0.1 & 0.2 & -0.4 & 0 \\ 1 & 1 & 1 & 100 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -0.5 & -1 & 0 \\ 0 & -0.25 & 0.3 & 0 \\ 0 & 0.25 & -0.3 & 0 \\ 0 & 1.5 & 2 & 100 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -1.6 & 0 \\ 0 & 1 & -1.2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8 & 100 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 42.105 \dots \\ 0 & 1 & 0 & 31.578 \dots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 26.315 \dots \end{bmatrix}$$

Where the final matrix gives us the percentage distribution across a , b and c .

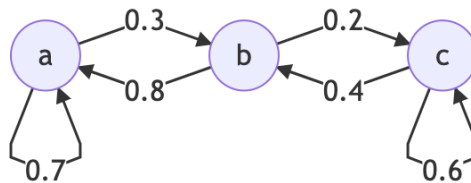
Notice that you can see our solution from above in the upper portion of the second last matrix.

© Example 3: Matrix→Diagram, Equilibrium

Leaving out intermediate steps now. Draw the diagram and find the equilibrium state for the transition matrix:

$$\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.8 & 0 & 0.2 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

Diagram:



Equilibrium:

$$\begin{aligned} 0.7a + 0.8b &= a &\rightarrow & -0.3a + 0.8b = 0 \\ 0.3a + 0.4c &= b &\rightarrow & 0.3a - b + 0.4c = 0 \\ 0.2b + 0.6c &= c &\rightarrow & 0.2b - 0.4c = 0 \end{aligned}$$

Solving the system of equations yields,

$$\begin{aligned} a - \frac{16}{3}c &= 0 &\rightarrow & a = \frac{16}{3}c \\ b - 2c &= 0 &\rightarrow & b = 2c \\ 0 &= 0 \end{aligned}$$

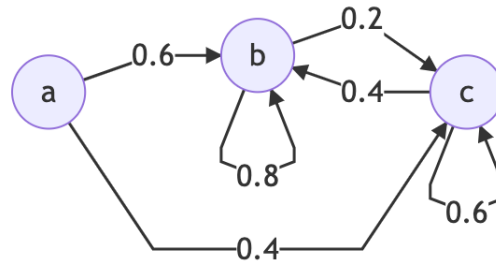
Or in terms of percentages, $a = 64\%$ $b = 24\%$ $c = 12\%$

What does Excel say happens? Starting from an initial even distribution across a , b and c we get,

			a	b	c		a	b	c		a	b	c		a	b	c		a	b	c		a	b	c
		a	0.7	0.3	0.0		0.7	0.3	0.0		0.7	0.3	0.0		0.7	0.3	0.0		0.7	0.3	0.0		0.7	0.3	0.0
		b	0.8	0.0	0.2		0.8	0.0	0.2		0.8	0.0	0.2		0.8	0.0	0.2		0.8	0.0	0.2		0.8	0.0	0.2
		c	0.0	0.4	0.6		0.0	0.4	0.6		0.0	0.4	0.6		0.0	0.4	0.6		0.0	0.4	0.6		0.0	0.4	0.6
a	b	c	a	b	c		a	b	c		a	b	c		a	b	c		a	b	c		a	b	c
33	33	33	50	23	27		54	26	21		58	24	18		60	24	15		62	24	14		63	24	13

⊙ Example 4: Diagram → Matrix, Equilibrium

One last example, again leaving out the intermediate steps. Find the transition matrix and the equilibrium state for the system:



Transition matrix:

$$\begin{array}{c} a \quad b \quad c \\ \begin{array}{l} a \\ b \\ c \end{array} \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0 & 0.8 & 0.2 \\ 0 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Equilibrium. Solve,

$$\begin{aligned} 0 + 0 + 0 &= a \\ 0.6a + 0.8b + 0.4c &= b \\ 0.4a + 0.2b + 0.6c &= c \end{aligned}$$

The first equation tells us $a = 0$ which simplifies the next two equations to,

$$\begin{aligned} 0.8b + 0.4c &= b &\rightarrow &-0.2b + 0.4c = 0 \\ 0.2b + 0.6c &= c &\rightarrow &0.2b - 0.4c = 0 \end{aligned}$$

Or $b = 2c$, which like our first example means $\sim 67\%$ b and $\sim 33\%$ c at equilibrium.